How Math Models the Real World

(and how it does not)

Saleem Watson
California State University, Long Beach
saleem@csulb.edu

How Math Models the Real World

"Traditional College Algebra is a boring, archaic, torturous course that does not help students solve problems or become better citizens. It turns off students and discourages them from seeking more mathematics learning."

Q: Who said this?

A: Chris Arney,

Dean of Science and Mathematics, St. Rose College

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Rethinking the courses below calculus

- NSF conference on “Rethinking the Courses Below Calculus” in Washington D.C in 2001. Some of the major themes to emerge from this conference:
- Spend less time on algebraic manipulation and more time on exploring concepts
- Reduce the number of topics but study those topics covered in greater depth
- Emphasize the verbal, numerical, graphical and symbolic representations of mathematical concepts
- Give greater priority to data analysis as a foundation for mathematical modeling

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Why mathematical modeling?

Virtually any educated individual will need the ability to:

1. Examine a set of data and recognize a behavioral pattern in it.
2. Assess how well a given model matches the data.
3. Recognize the limitations in the model.
4. Use the model to draw appropriate conclusions.
5. Answer appropriate questions about the phenomenon being studied.

- Sheldon Gordon, Farmingdale State University of New York

- Provides an answer to the question: Why study math?

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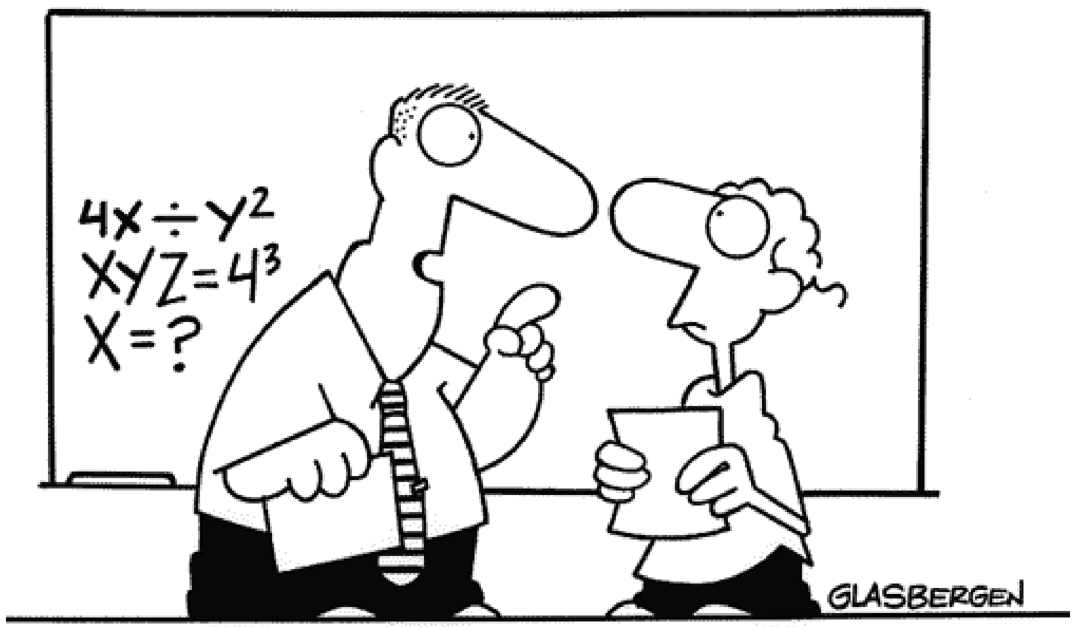

"Algebra class will be important to you later in life because there's going to be a test six weeks from now."

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What is a mathematical model?

- A model is a mathematical description of a real-world situation. Generally describes only one aspect of the real-world situation
- A model must allow us to make predications about the thing being modeled.
- Most of the models we construct in lower division courses are functions.
- For a function to actually model a real world situation, the independent variable of the function must reasonably "predict" the dependent variable.

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For example . . .

- A street map of a city is a model of the city; the map models the relative location of the city streets to each other.

- When driving, a street map helps us to predict the location of an upcoming intersection.
- If a map does not help us navigate a city, then that map is not a model—it is useless.

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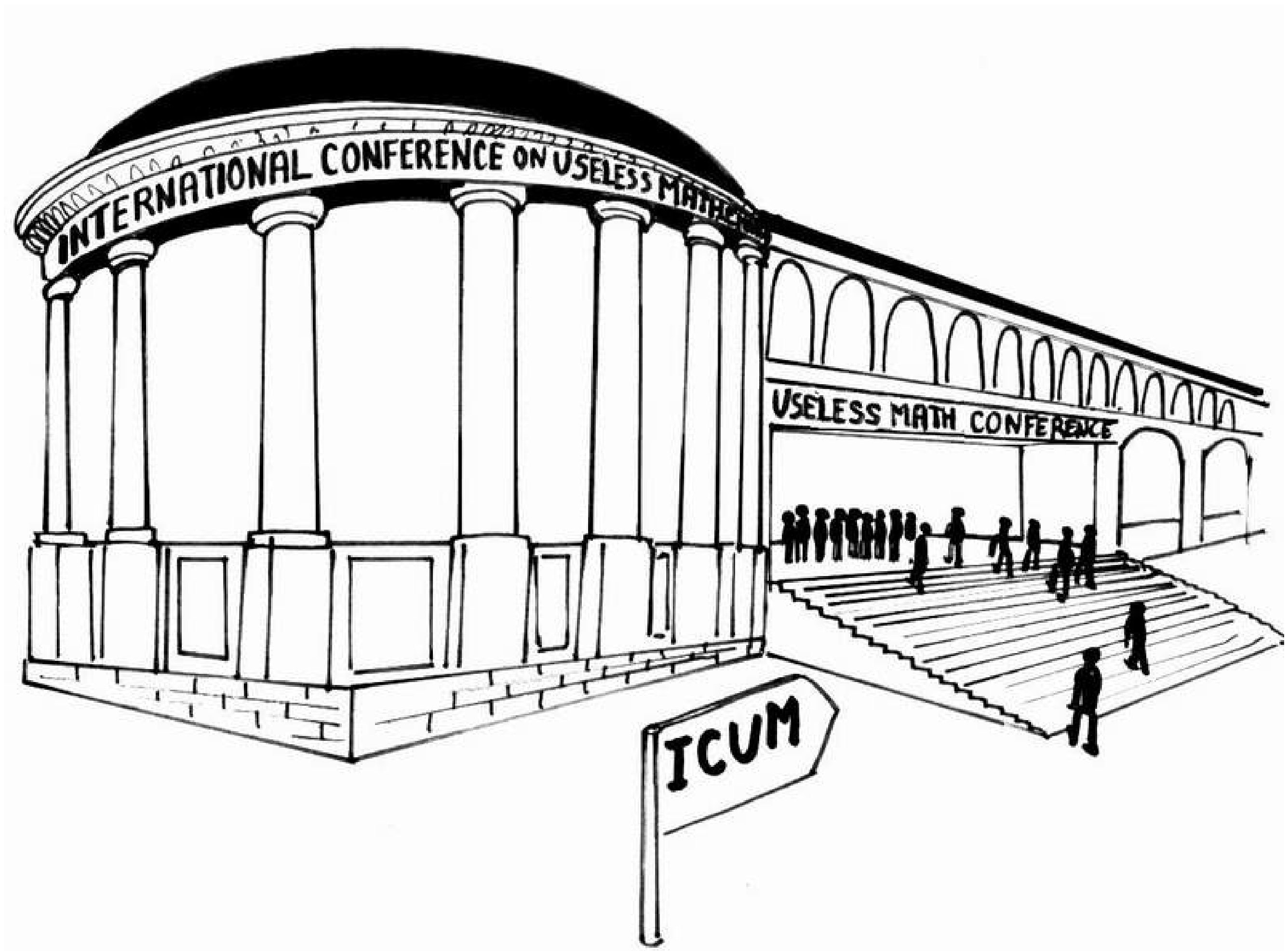

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How to construct a model from data

- Start with a set of real-world data
- Look for a pattern in the data.
- Describe a principle that produces the data.
- Try to express what we've discovered about the data algebraically, as a function.
- The model allows us to make conclusions about the thing being modeled. The independent variable of the function "predicts" the dependent variable

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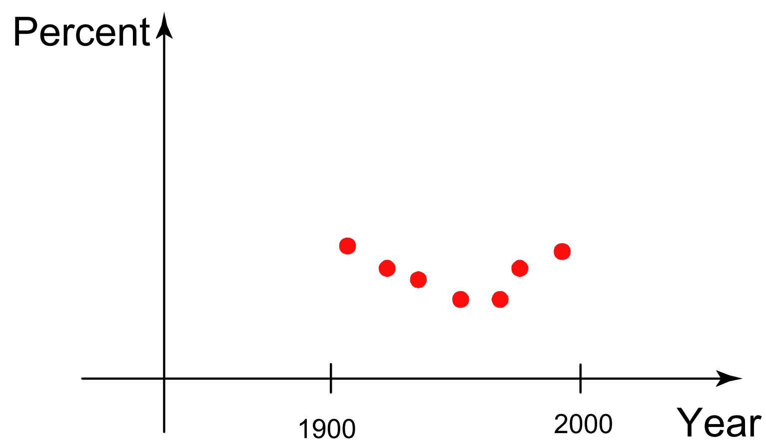
Example (from a textbook)

- Data for percent of US population that is foreign born *vs* year
- We are asked to find a *quadratic function* that "models" the data.

Year	Percent
1900	13.6
1920	13.2
1940	8.8
1960	5.4
1900	6.2
2000	11.1



Graph of data

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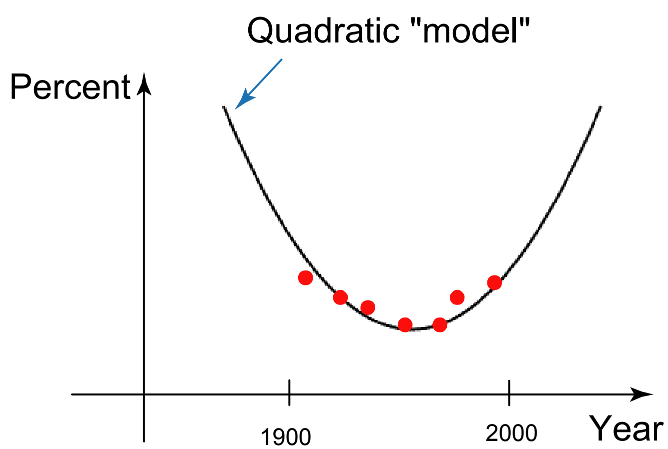
Example (from a textbook)

- Data for percent of US population that is foreign born *vs* year
- We are asked to find a *quadratic function* that "models" the data.



Graph of data and "model"

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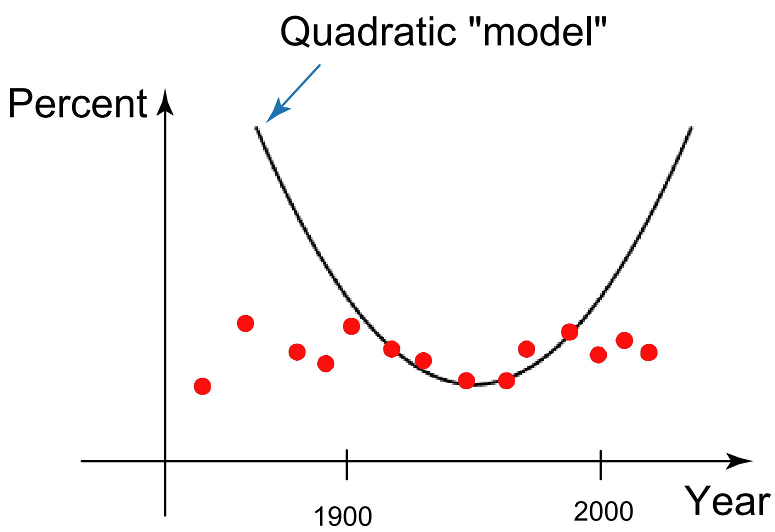
Example (from a textbook)

- Data for percent of US population that is foreign born *vs* year
- We are asked to find a *quadratic function* that “models” the data.



No prediction, NOT a model

Is there any real-world purpose to such an example?

Does this “model” have any use?

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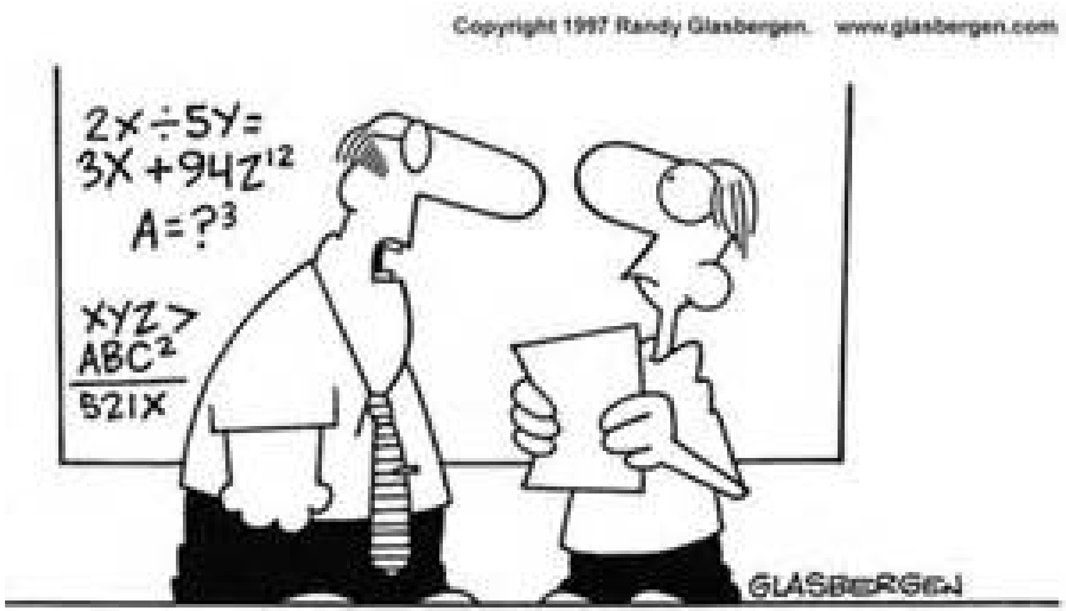

“Why is it important for today’s kids to learn algebra? Because *I* had to learn this junk in school and now it’s *your* turn, that’s why!”

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What went wrong in this example?

- Does the curve fit the data well?
- Why a quadratic function?
- Do other types of functions also fit the data?
- Is there a principle that generates these data?
- Does the quadratic function produced predict the number of foreign born citizens?

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More examples (from textbooks)

- Number of families on social assistance vs year (quadratic)
- Number of larceny thefts vs year (cubic)
- CD sales vs year (cubic)
- Value of imported goods vs year (cubic)

❑ In each case we can ask: How does the year cause a change in the quantity?

  - Are these really models? Are they of any use?

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"Algebra will be useful to you later in life because it teaches you to shut up and accept things that seem pointless and stupid."

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Curve Fitting

Given a set of two-variable data

- Find the line that best fits the data.
  - This is the line with the property that the sum of the squares of the distances from the line to each data point is minimized.
  - Finding the line of best fit is a purely mathematical procedure that is not related to any real-world meaning that the data may have.

- For a given set of data, we can find
  - The linear function that best fits the data
  - The quadratic function that best fits the data
  - The cubic function that best fits the data
  - The exponential function that best fits the data
  - Etc.

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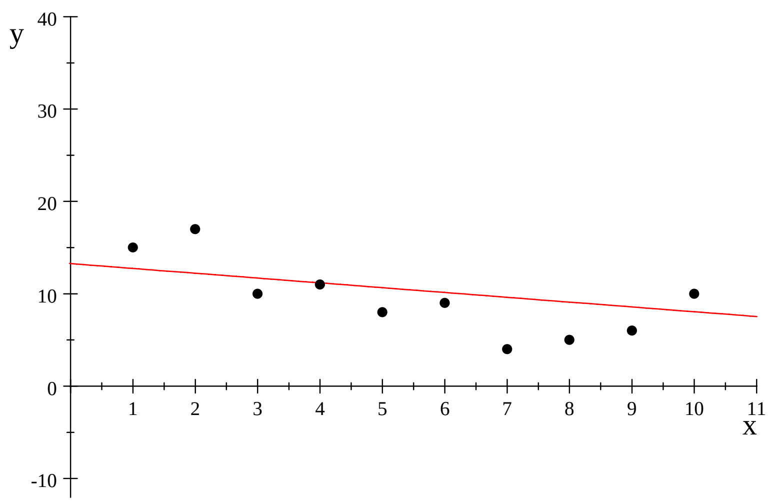
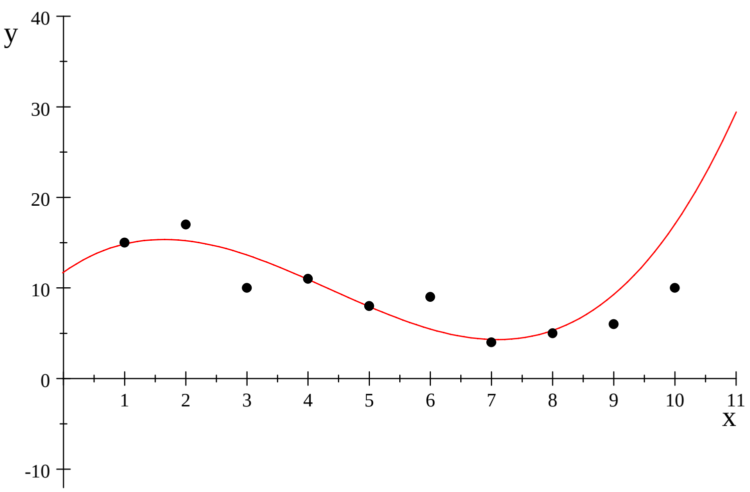
Curve Fitting

- The graphs below show the line of best fit and the cubic polynomial of best fit for the data in the above example.





Linear function

Cubic function

- Is the curve that best fits the data a model?

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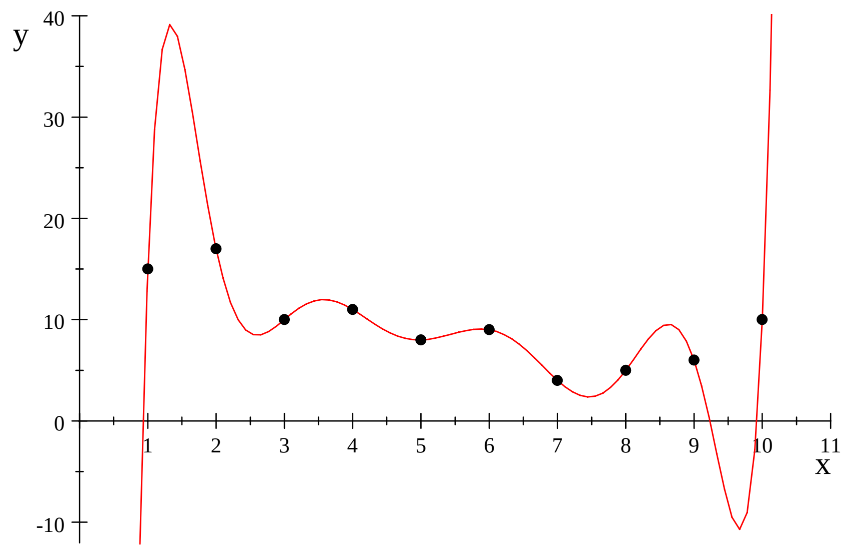
Curve Fitting

- Best of all: Given *n*+1 data points in general position, we can find a polynomial function of degree *n* that fits the points exactly (that is, passes through all the points).



Degree 9 polynomial

- Is this then the best "model" for these data?

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Curve Fitting versus Modeling

- Fitting a curve to data is not modeling
- The art of modeling is largely about what type of function is appropriate for a particular set of real-world data.
- Fitting a curve to data is a purely mathematical process that is not related to any real-world meaning the data may have.
- Curve fitting is often confused with modeling
- A bad "model" implies that math is of no use, or perhaps good for one thing,. . .

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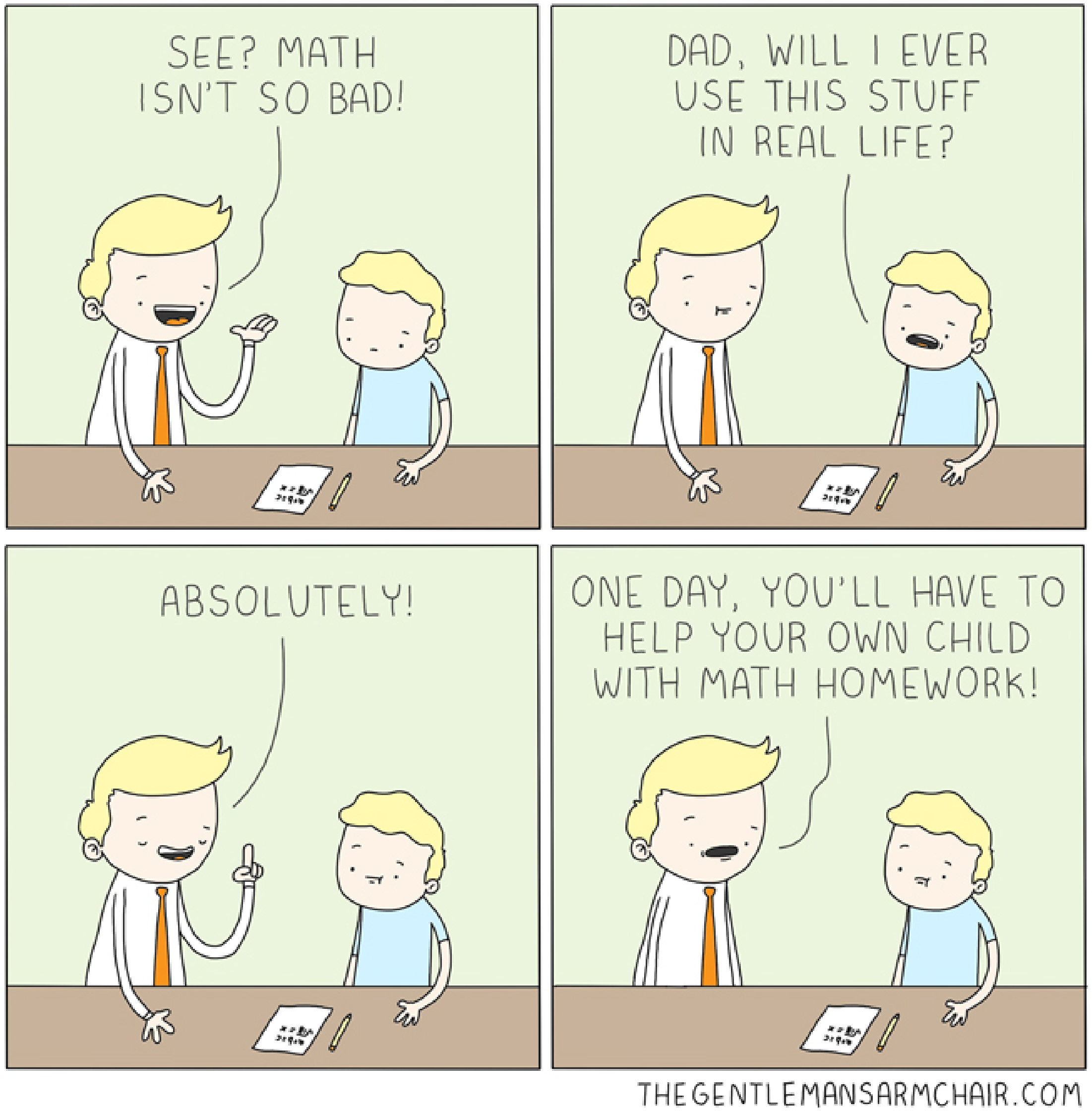

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Finding a Model for Real-World Data

PART I Reasoning about a process

- Modeling data on the population of an animal species.
- Reasoning: What causes the population to increase?
- Suppose that the average number of offspring for each individual in the population is about 3. Then a single individual would increase to 3, then 9, then 27, …
- Principle: Population growth is exponential. So we model these data by an *exponential function*.
- A graphing calculator allows us to find the *exponential function* that best fits the data.

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Example

- **Data**: for an animal population
- **Principle**: growth is exponential
- **Model**: The exponential function that best fits the data
- **Graph**: just to “see” the growth and to check the “goodness of fit” of the model.
- **Prediction**: What is the expected population at the next stage?

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Example

x	*P*
1	2
2	9
3	27
4	83
5	240

Principle: Exponential

Model

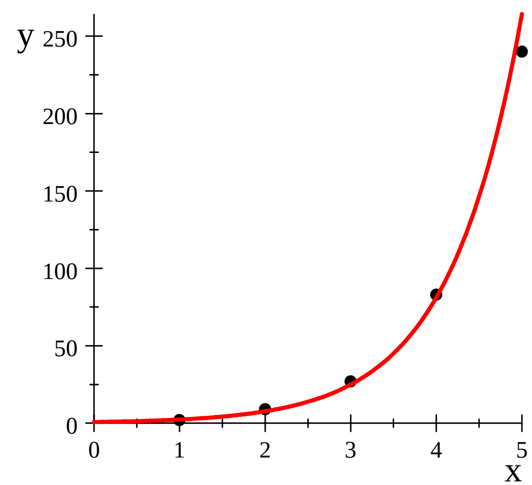
$$P(x) = 0.7248(3.253)^x$$

(using calculator)

Prediction

$$P(6) = 0.7248(3.253)^6$$

$$\approx 859$$



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Reasoning: What type of function model?

- Surface area of small lakes in Ontario, Canada versus average width of each lake.
  - quadratic
- Height of a mountain versus its volume
  - cubic
- Height of skyscraper versus the number of floors.
  - linear
- Speed of a car versus gas mileage at that speed.
  - quadratic
- Ocean depth versus water pressure.
  - Linear
- The distance a cannonball falls from the leaning Tower of Pisa versus the time it has been falling
  - quadratic

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Reasoning: What type of function model?

- Tire inflation versus tire life
  - quadratic
- Amount of rainfall versus crop yield
  - quadratic
- Area-species relationship
  - power
- How quickly can you list your favorite things?
  - cubic
- How quickly does water leak from a tank
  - Quadratic

❑ In each example the independent variable of the model allows us to "predict" the dependent variable.

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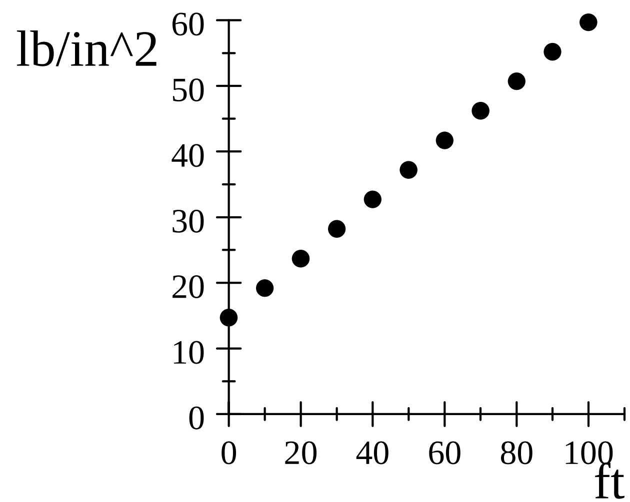
Example: Pressure versus Depth

William Beebe
With Bathysphere

Depth (ft)	Pressure (lb/ft²)
0	14.7
10	19.2
20	23.7
30	28.2
40	32.7
50	37.2



Linear function model: $y = 14.7 + 0.45x$

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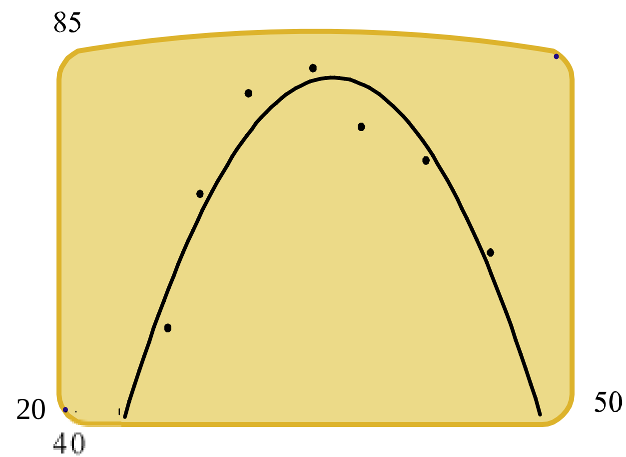
Example: Tire inflation-tire life relation

– Quadratic functions

Tire Pressure/Tire Life

Pressure (lb/in^2)	Tire life (mi X1000)
26	50
28	66
31	78
35	81
38	74
42	70
45	59

QuadReg
y=ax ^{2}+bx+c
a=-.24324
b= 17.627
c= 239.47



Model: $y = -0.24324x^2 + 17.627x - 239.47$

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Example: Length-at-age relation

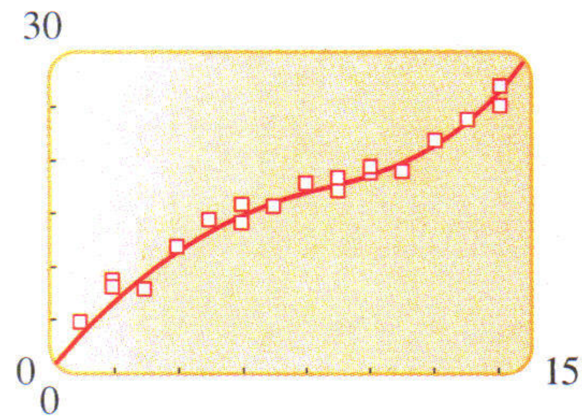
– cubic

90 year old rock fish

Length-at-age data

Age (years)	Length (inches)
1	4.8
2	8.8
2	8.0
3	7.9
4	11.9
5	14.4
6	14.1
6	15.8
7	15.6
8	17.8

Ag e (ye ars)	Length (inches)
9	18.2
9	17.1
10	18.8
10	19.5
11	18.9
12	21.7
12	21.9
13	23.8
14	26.9
14	25.1



Model: $y = 0.0155x^3 - 0.372x^2 + 3.95x + 1.21$

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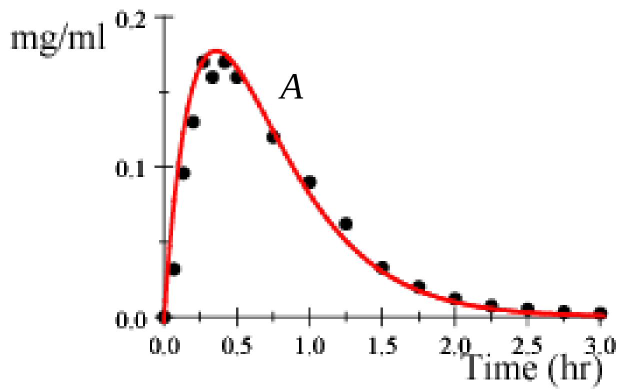
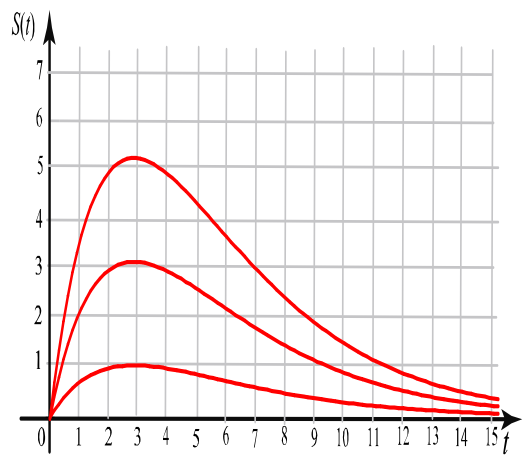
Example: Alcohol level versus hours since consumption

– Surge Functions



$$S(t) = at \cdot b^t$$

$$(a > 0,\ 0 < b < 1)$$



Concentration (mg/ml) after 95% ethanol oral dose

Time (hr)	15 ml	30 ml	45 ml	60 ml
0.	0.	0.	0.	0.
0.067	0.032	0.071	—	—
0.133	0.096	0.019	—	—
0.167	—	—	0.28	0.30
0.2	0.13	0.25	—	—
0.267	0.17	0.30	—	—
0.333	0.16	0.31	0.42	0.46
0.417	0.17	—	—	—
0.5	0.16	0.41	0.51	0.59
0.667	—	—	0.61	0.66
0.667	—	—	0.61	0.66
0.667	—	—	0.61	0.66
0.667	—	—	0.61	0.66
⋮	⋮	⋮	⋮	⋮

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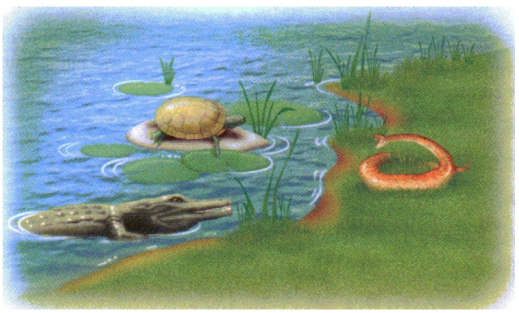
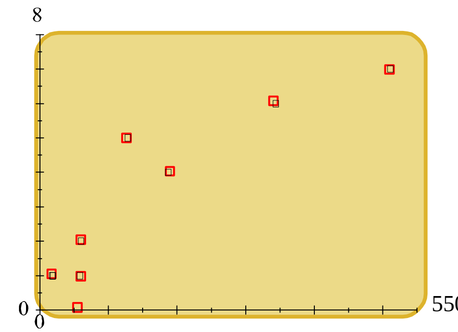
Example: Species-Area Relation

– Power functions

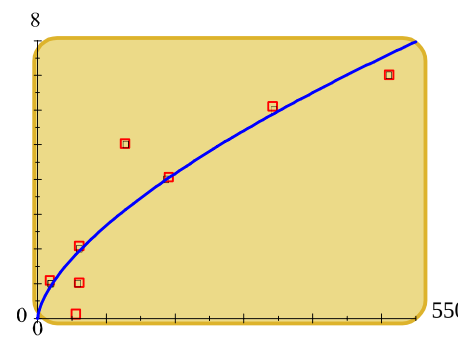


Species-area data

Cave	Area (m^2)	Number of species
La Escondida	18	1
El Escorpion	19	1
El Tigre	58	1
Mision Imposible	60	2
San Martin	128	5
El Arenal	187	4
La Ciudad	344	6
Virgen	511	7



PwrReg
y = a * x ^ b
a = 0.140019
b = 0.640512



Model: $y = 0.14x^{0.64}$

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Simple classroom experiments

- Bridge science

Layers	Pennies
1	4
2	10
3	16
4	22
5	28
7	**Predict**
Predict	**50**

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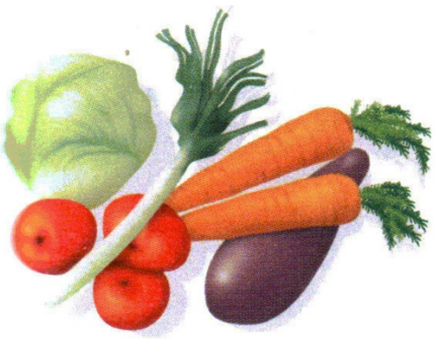
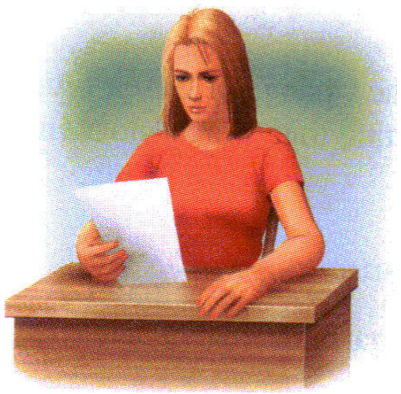
Simple classroom experiments

- How quickly can you name your favorite things?
- How long can you remember a memorized list?



Listing vegetables



Memorizing a list

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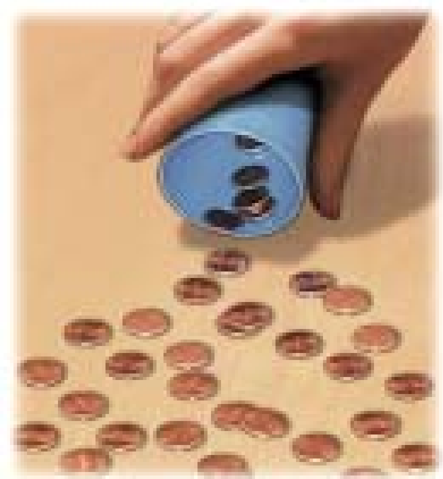
Simple classroom experiments

- Radioactive decay—modeled with pennies

Radioactive Decay



Coin Experiment

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Simple classroom experiments

– How quickly does water leak from a tank? Toricelli's Law

Toricelli's Law

The experiment

Students at Tennesee Tech performing the experiment

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Reasoning: What type of function model?

- Time since the first reported case of a viral infection versus the number of persons infected with the virus.
  - ➊???????
- The number of tumors in laboratory rats versus the amount of exposure to asbestos
  - ➋???????
- The value of the NASDAQ versus the date in 2015.
  - ➌???????

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Finding a Model for Real-World Data

Part II Reasoning with calculus about a process

- Modeling data on the population of an animal species.
- Principle: (by reasoning about rates of change)

  "The rate of change of population is proportional to the population.

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- Calculus: express the principle as a differential equation
  - Differential equation:

$$\frac{dP}{dt} = kP, \quad P(0) = P_0$$

  - Solution to differential equation:

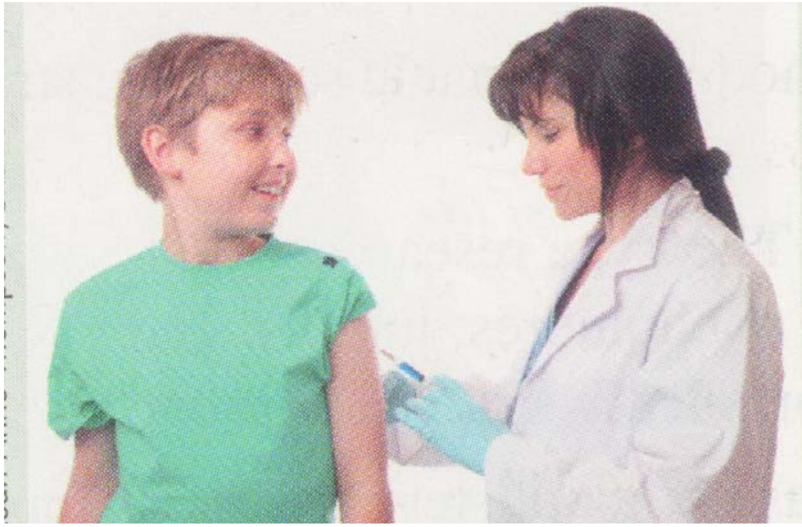
$$P(t) = P_0 e^{kt}$$

- Principle: (from the solution to the differential equation)

  Population growth is exponential. So we model these data by an *exponential function*.

❶ Example

Number of persons infected with a virus versus the time since the first reported case.



➊ Example

Number of persons infected with a virus versus the time since the first reported case.

- Principle: (by reasoning about rates of change)

  "The rate of change of the number of infected individuals is jointly proportional to the number of infected and noninfected individuals."

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- Calculus: express the principle as a differential equation
  - Differential equation:

$$\frac{dA}{dt} = kA(N - A), \quad A(0) = A_0$$

  - Solution to differential equation:

$$A(t) = \frac{N}{1 + \left(\frac{N}{A_0} - 1\right)e^{-kt}}$$

- Principle: (from the solution to the differential equation)
  - Spread of disease is described by this type of function (logistic function).

$$y = \frac{c}{1 + ae^{-bt}}$$

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A city of 50,000 initially has 100 cases of a viral flu, 1000 people were infected after ten weeks.

(a) Find a model.

$$A(t) = \frac{50,000}{1 + 499e^{0.23208t}}$$

(a) Predict the number of infections after 15 weeks

$$A(15) = \frac{50,000}{1 + 499e^{0.23208(15)}} \approx 3057$$

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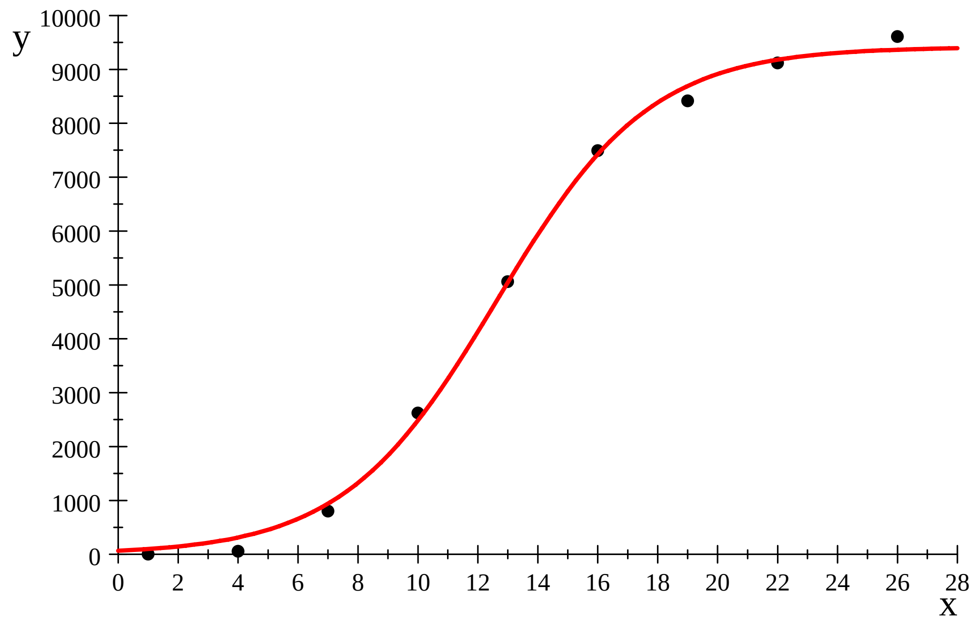
Is a logistic model reasonable?

The table gives the reported number of AIDS cases in South Africa.

Year	Number
1981	2
1984	55
1987	802
1990	2623
1993	5060
1996	7493
1999	8416
2002	9118
2005	9609



Logistic Model: $y = \dfrac{9416}{1 + 138.5e^{-0.3904x}}$

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We don’t always have a principle

- For many real-world data we do not *a priori* know the principle that generates the data.
- We can use the data to discover the underlying principle.
- This type of modeling problem is vastly more difficult.
- For example,
  - Galileo discovered that the distance that an object falls is related to the square of the time it has been falling, using experimental data
  - Kepler discovered the planetary laws of motion from data

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Models are a work in progress

- A model is never perfect
- The model can always be improved to conform to real-world realities
- For example:
  - Falling object: $y'' = -g$
  - Falling object with air resistance: $y'' = -g + ky'$
  - Taking shape of object into account: fluid dynamics
  - Many other factors.
- The assumptions under which a model is constructed should be completely stated.

Finding a Model for Real-World Data

PART III Reasoning using statistics

- There are many modeling situations where the data are not so predictable.
- In such cases we can use statistics to discover trends in the data. Such trends model one aspect of the phenomenon being studied.
- For example, …

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How do we model the following situation?

- The number of tumors laboratory rats develop versus the level of exposure to asbestos.
  - Is a linear, quadratic, exponential or some other type of model appropriate?
  - This can be presented to students as a research topic:
    - Obtain data
    - Try different models
    - Which type of model fits best?

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The central question about asbestos/tumor data is as follows

- Does an increase in asbestos levels result in a significant increase in the number of tumors?
- If there is a linear trend in the data, is it likely just a result of chance?
- Is there a significant correlation between asbestos levels and the number of tumors?

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- To model this situation
  - Find the regression line an correlation coefficient.
  - The correlation coefficient ($-1 \le r \le 1$) measures how well the data fit the regression line.
  - The correlation coefficient does not tell us whether the regression line is a good model for the data.

- We need to answer the following
  - Is the correlation we obtained significant at the 0.05 significance level?
  - The significance of the correlation is determined by a *t*-test that involves *r* and *n* (the number of data points).
  - *t*-test

$$t_{n-2} = \sqrt{n-2}\,\frac{r}{\sqrt{1-r^2}}$$

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- The meaning of the *t*-test
  - Assumption: The data are a random sample from a population with $\rho = 0$
  - A *t*-test gives the probability of picking a random sample from this population with correlation *r.*
  - If the *t*-test gives a *P*-value less than 0.05 (the significance level) we say that there is a statistically significant correlation between tumors and exposure to asbestos.

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❷ Example

Exposure to asbestos versus number of tumors

Asbestos fibers	Number of tumors
50	2
400	6
500	5
900	10
1100	26
1600	42
1800	37
2000	28
3000	50

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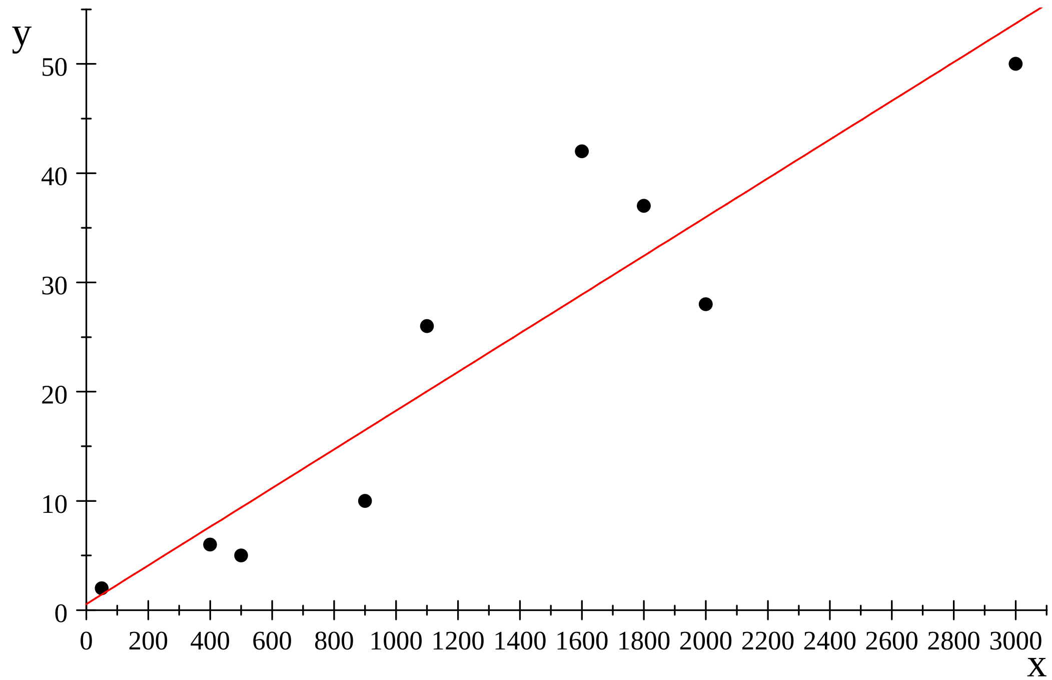
❷ Example

Exposure to asbestos versus number of tumors

Asbestos fibers	Number of tumors
50	2
400	6
500	5
900	10
1100	26
1600	42
1800	37
2000	28
3000	50



Regression line $y = 0.017x + 0.54$

Correlation coefficient: r=0.92

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From the table a correlation coefficient of r=0.92 with 9 data points, is significant at the 0.05 significance level.

Critical values of the correlation coefficient *r* for sample size *n* (at the 95% confidence level)

n	r	n	r	n	r
3	0.997	11	0.602	35	0.334
4	0.950	12	0.576	40	0.312
5	0.878	13	0.553	50	0.279
6	0.811	14	0.532	60	0.254
7	0.754	15	0.514	70	0.235
8	0.707	20	0.444	80	0.220
9	0.666	25	0.396	90	0.207
10	0.632	30	0.361	100	0.197

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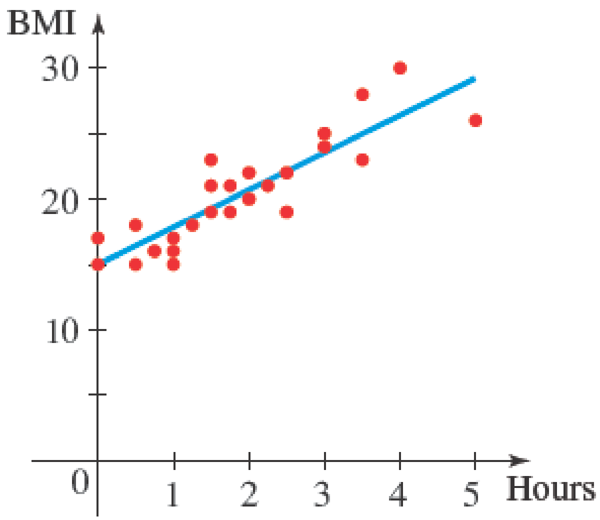
Example

TV Hours/BMI

Hours TV	BMI
0	15
0	17
.5	15
.5	18
.75	16
1	16
1	15
1	17
1.25	18
1.5	19
⋮	⋮



Regression line $y = 2.1x + 14.2$

Correlation coefficient: $r = 0.88$ (significant)

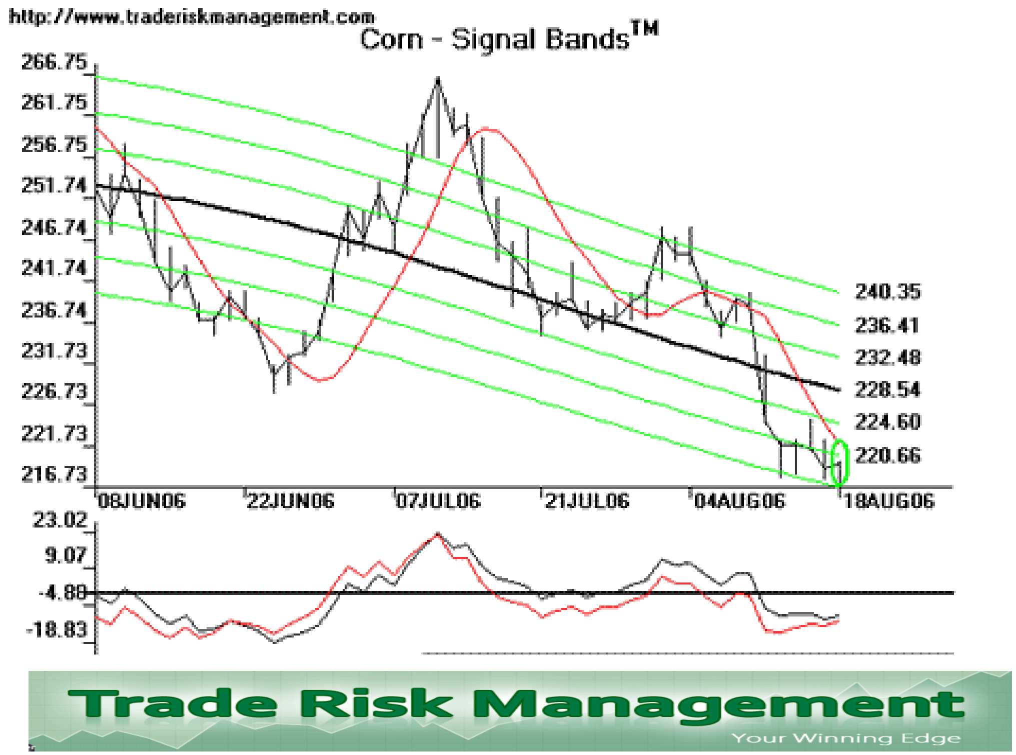
How Math Models the Real World

Are there consequences to bad "models"?

- What if William Bebe had used a bad model for the pressure/depth data?

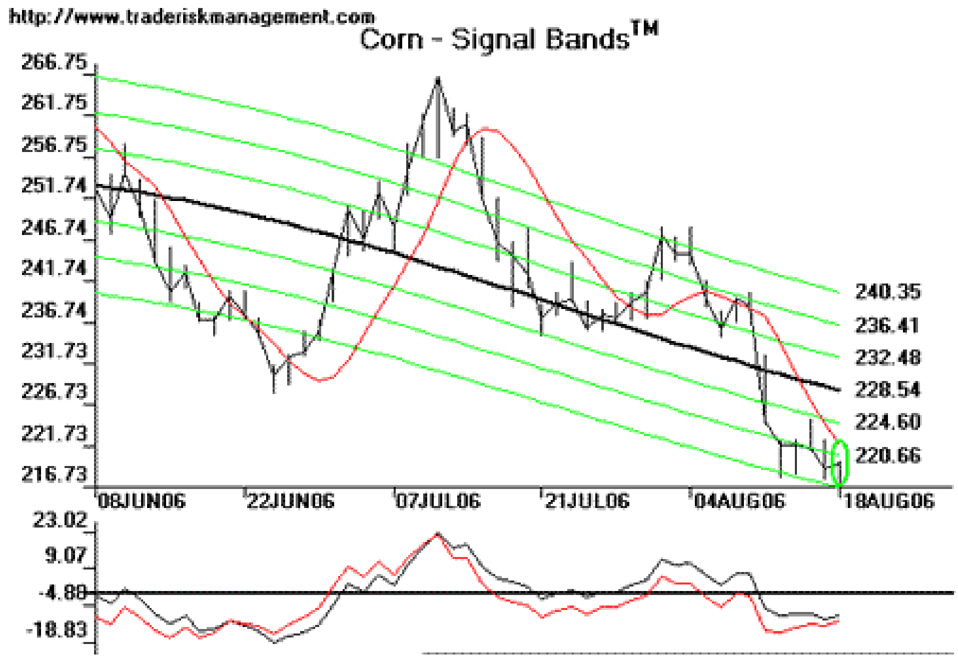
How Math Models the Real World

❸ Example A company called Trade Risk Management was selling "models" of the stock market as aids to investors.



How Math Models the Real World

- The independent variable is time.
- Can the time predict the value of a stock?



Trade Risk Management
Your Winning Edge

How Math Models the Real World

Real world result:

Trade Risk Management

Your Winning Edge

Indicted for Fraud

by the CFTC

Release: 5142-05

For Release: December 7, 2005

Washington State Commodity Advisory Firm Trade Risk Management and Firm President xxx xxx Charged With Fraud

Washington, DC – The Commodity Futures Trading Commission (CFTC) today announced that it filed an anti-fraud enforcement action in the United States District Court for the Western District of Washington at Tacoma against **Trade Risk Management, LLC** (Trade Risk) and **xxxx xxxx** of Vancouver, Washington.

The CFTC complaint, filed on November 25, 2005, alleges that xxx and Trade Risk fraudulently solicited customers to purchase a futures charting service known as **Sigma Band Charting** (Sigma Band) through the Internet website www.traderiskmanagement.com by falsely claiming that, among other things, use of the Sigma Band charts would give customers a 99 percent chance of making money every time they traded. xxx and Trade Risk allegedly attracted over 420 subscribers, and the firm collected approximately $400,000 in customer fees.

The CFTC is seeking a permanent injunction against xxxxx and Trade Risk, repayment to defrauded customers, the return of all ill-gotten gains from the defendants, and monetary penalties.

Conclusion

- To be considered a real-world model, a function must reasonably predict (real-world) values of the dependent variable from the corresponding (real-world) values of the independent variable.
- Modeling real-world data is a complex process that should not be confused with fitting a curve to data.
- For some data a clear principle suggests how the data is produced, and thus indicates the type of function that is appropriate for modeling.
- For some situations we can use calculus to discover the type of function needed for modeling.

How Math Models the Real World

Conclusion

- For some data the process of modeling involves discovering the appropriate type of function needed to model the data. This may not always be achievable, but …

- Linear models together with a statistical analysis of the data (including the correlation coefficient and the number of data points) may help answer critical questions about the real-world situation represented by the data.

- A model is never perfect; it is always a work in progress

- We can do simple classroom experiments to try out for ourselves the entire process of modeling and prediction for a real-world situation.

THANK YOU